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Trigonometric Sequences and Series

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Trigonometric Sequences and Series

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Abstract

Trigonometric Sequences and Series

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This report discusses the background of trigonometric sequences and series related to defining the sine and cosine functions. Proofs involving converging trigonometric sequences and series are presented using nontraditional methods. To conclude, an application of trigonometric sequences and series is shown.

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Chapter 1: Introduction

In Precalculus, students first experience trigonometric functions as well as arithmetic and geometric sequences and series. Students first consider the concepts of limits and series convergence. Once in Calculus, students learn more about limits, derivatives, and integrals. Although trigonometric sequences and series are discussed, the concept is unfortunately not taught in depth. In this report, trigonometric sequences and series are portrayed at a level a Calculus student can understand. Techniques and proofs involving trigonometric sequences and series are explored using nontraditional approaches.

The discussion begins with Sholander defining sine and cosine, the two most common trigonometric functions, using Fibonacci trigonometric double sequences [**5**]. Sholander uses the Fibonacci trigonometric double sequences to portray the four main properties of sine and cosine: being cosinusoidal over real numbers, having uniform continuity, having a period of 2π, and being monotonic in the first quadrant.

The convergence and divergence of series are then considered. By exploring tangent sequences, Rosenholtz presents three main results about tangent series [**4**]. In order to prove the three results, Rosenholtz uses number theory, several million digits of π, the mean value theorem, and other approaches.

The topic of Laplace transforms is subsequently compared to Fourier analysis, which is known for finding exact values of convergent series. Efthimiou believes Laplace transforms is the better method because it requires less guesswork and is more straightforward [**1**].

Further, an application of using trigonometric sequences and series is shown through approximating the digits of π to over 300,000 digits. Kreminski discovers an accelerated version of Vieta's formula that accurately approximates π through number crunching and experimentation [**2**].

Chapter 2: Defining Sine and Cosine

Sine and cosine are often defined using power series, differential equations, or definite integrals. However, Sholander defines sine and cosine at a more elementary level through the use of Fibonacci sequences, specifically Fibonacci trigonometric double sequences, otherwise known as FTD sequences [**5**]. The process includes nine theorems with a few lemmas and corollaries.

Sholander builds two functions that have four properties: both are cosinusoidal over the real numbers, are uniformly continuous, have a period of 2π, and are monotonic in the first quadrant, $0 \le x \le \frac{\pi}{2}$. After showing the two functions have all four properties, the functions can be defined as sine and cosine.

Sholander begins by defining an *FTD sequence*. The set $\{a_0, a_1, ...; b_0, b_1, ...\}$ where a_i and b_i are real numbers is an FTD sequence if

$$a_0 = 1,\ b_0 = 0,\ {a_1}^2 + {b_1}^2 = 1,$$

$$a_{n+1} = 2a_1 a_n - a_{n-1},$$

$$b_{n+1} = 2a_1 b_n - b_{n-1}\ . \qquad [\mathbf{5}, \text{p. } 73]$$

Theorem 1 states:

An FTD sequence is cosinusoidal over the nonnegative integers, i.e., it admits the identities

$${a_k}^2 + {b_k}^2 = 1,$$

$$a_{n+k} = a_n a_k - b_n b_k ,$$

$$b_{n+k} = b_n a_k + a_n b_k \ . \qquad [\mathbf{5}, \text{p. } 73]$$

Since an FTD sequence is cosinusoidal over the nonnegative integers, other identities follow using trigonometry. Sholander's Corollary expresses the following identities:

$$a_{n-k} = a_n a_k - b_n b_k ,$$

$$b_{n-k} = b_n a_k - a_n b_k ,$$

$$a_{n+k} + a_{n-k} = 2a_n a_k ,$$

$$a_{n+k} - a_{n-k} = -2b_n b_k ,$$

$$b_{n+k} + b_{n-k} = 2b_n a_k ,$$

$$b_{n+k} - b_{n-k} = 2a_n b_k ,$$

$$a_2 + a_4 + ... + a_{2n} = a_{n+1} b_n / b_1 . \qquad [\mathbf{5}, \text{p. } 73]$$

Using the identities from Theorem 1 and the Corollary, Theorem 2 is proven. Theorem 2 states:

> If FTD sequence $\{a_0, ...; b_0, ...\}$ has bisection $\{A_0, ...; B_0, ...\}$ then $A_{2n} = a_n$ and $B_{2n} = b_n$. If the former has (minimal) period $p = 2^m$, so $a_{n+p} = a_n$ and $b_{n+p} = b_n$, then the latter has period $2p$." [**5**, p. 74]

where a_i, b_i, A_i, and B_i are real numbers.

Sholander then considers a special case, where

$$S_1 = \{1, -1, 1, -1, ... ; 0, 0, ...\},$$

$$S_2 = \{1, 0, -1, ... ; 0, 1, 0, ...\},$$

$$S_3 = \{1, \tfrac{1}{\sqrt{2}}, 0, ... ; 0, \tfrac{1}{\sqrt{2}}, 1, ...\}, \ldots \qquad [\mathbf{5}, \text{p. } 74]$$

Sholander states Theorem 3 using the special case. Theorem 3 expresses a_n is decreasing from 1 to 0 in the first quadrant of S_n, while b_n is increasing from 0 to 1. After proving Theorem 3 using induction, Sholander expands "S_n to T_n, a two-way FTD sequence, by defining $a_{-m} = a_m$ and $b_{-m} = b_m$" [**5**, p. 74].

Sholander's Theorem 4 shows T_n is cosinusoidal over the integers by using the new two-way FTD sequence T_n. The first property to define sine and cosine states the two functions are cosinusoidal over the real numbers. Thus, Sholander expands integers to the set of *dyadic fractions*, which is a rational number whose denominator has a power of two.

Sholander defines two functions $C(r)$ and $S(r)$ as terms in T_n for some fixed natural number *n* sufficiently large and integer *m*, where $r = \frac{m}{2^n}$ [**5**, p. 74]. Next, Theorem 5 states the "functions $C(r)$ and $S(r)$ are cosinusoidal over the set of dyadic fractions. Both have period 1. In the first quadrant, $C(r)$ decreases from 1 to 0 and $S(r)$ increases from 0 to 1" [**5**, p. 74].

In addition, Sholander proves another Corollary showing that $C(r)$ and $S(r)$ are uniformly continuous over the dyadic fractions [**5**, p. 74]. Because of this Corollary, the functions can be extended to the real domain. Sholander states in Theorem 7 that $C(r)$ and $S(r)$ inherit the properties of Theorem 5 and 6. Therefore, Sholander has shown three of the four properties. The last property of the functions having a period of 2π is demonstrated as Theorem 8.

Sholander does not take the traditional route of defining sine and cosine but instead uses FTD sequences. Last, Theorem 9 sums up Sholander's definition of sine and cosine using the previously listed four properties.

Chapter 3: Tangent Sequences and Series

Many factors contribute to the determination of whether a series converges or diverges. One series that is not frequently mentioned in Calculus textbooks is the tangent series. Rosenholtz first explores tangent sequences and then specifically discusses whether each of the following tangent series converges

$$\sum_{n=1}^{\infty} \frac{\tan(n)}{n}, \ \sum_{n=1}^{\infty} \frac{\tan(n)}{n^2}, \ \sum_{n=1}^{\infty} \frac{\tan(n)}{n^3}, \ldots \tag{1}$$

[**4**, p. 367].

To prove the theorems and conjectures under investigation, Rosenholtz uses number theory and several million digits of π.

Considering convergence, Rosenholtz researches whether the terms of (1) approach zero. Rosenholtz finds three main results;

$$\lim_{n\to\infty} \frac{\tan(n)}{n} \text{ does not exist} \tag{2}$$

$$\lim_{n\to\infty} \frac{\tan(n)}{n^8} = 0 \tag{3}$$

$$\frac{|\tan(n)|}{n^2} \text{ is ``small''} \tag{4}$$

[**4**, p. 367].

Rosenholtz initially provides a background of knowledge about continued fractions expansion of an irrational number α. Specifically, the author provides an example of the first eight convergents for $\frac{\pi}{2}$. Rosenholtz then displays a gallery of facts about these specific types of continued fractions [**4**, p. 368].

Table 1. A Gallery of Facts

A Gallery of Facts About the Continued Fraction Expansion of an Irrational Number α
(a) $\alpha_1 = \alpha;\ a_n = [\alpha_n];\ \alpha_{n+1} = 1/(\alpha_n - a_n)$.
(b) $p_{k+1} = a_{k+1} p_k + p_{k-1};\ q_{k+1} = a_{k+1} q_k + q_{k-1}$.
(c) $p_1 < p_2 < p_3 < \ldots$ and $q_1 < q_2 < q_3 < \ldots$.
(d) $p_k q_{k+1} - p_{k+1} q_k = (-1)^k$.
(e) $\lvert p_k / q_k - \alpha \rvert < 1/q_k^2$.
(f) p_k / q_k $(k > 1)$ is the “best rational approximation” to α in the sense that any fraction that is closer to α than p_k / q_k must have larger denominator.
(g) If $\lvert p/q - \alpha \rvert < 1/2q^2$ with p/q reduced and $q > 0$, then, for some *k*, $p/q = p_k / q_k$; that is, p/q must be a convergent for α.
(h) $1/q_k(q_k + q_{k+1}) < \lvert p_k / q_k - \alpha \rvert < 1/q_k q_{k+1}$

where p_k and q_k are integers.

Rosenholtz proves (2) in two steps using contradiction. In the first step, Rosenholtz proves that if the limit of (2) exists, this limit must be 0 by constructing an increasing sequence. In the second step, using “Gallery of Facts” c, d, and e from Table 1, Rosenholtz shows the limit cannot be 0. Therefore, the limit does not exist.

Rosenholtz proves (3) by drawing on other mathematicians’ findings. Using Hata’s result and some algebra, the inequality

$$\left| \frac{q\pi}{2} - p \right| > \frac{1}{2q^{7.02}}$$

is formed [**4**, p. 370]. Two cases are formed when q is even or q is odd. If q is even, then (3) is proven instantaneously. If q is odd, application of the mean value theorem is needed to prove (3).

Rosenholtz works to prove (4) is true for all integers n. After proving Theorem 3 with the help of Lemma 1 and 2, Theorem 4, and Corollary 1, Rosenholtz found that "we need only look at numerators of those convergents of the continued fraction expansion of $\frac{\pi}{2}$ having odd denominators" [**4**, p. 374].

Therefore, Rosenholtz used digits of π and the computer algebra system (CAS) *Maple V* to find that $\frac{|\tan(n)|}{n^2}$ is small for all $n \leq 10^{60,000}$. Rosenholtz then found that $\frac{|\tan(n)|}{n^2}$ is small for all $n \leq 10^{8,000,000}$ using the CAS *Mathematica 3.0*. The results from *Mathematica 3.0*, Theorem 5, and Corollary 2 are shown in Table 2 [**4**, p. 375].

Table 2. Estimates for $|\tan(n)|/n^2$ for $n \le 10^{8,000,000}$

Estimates for $\lvert\tan(n)\rvert/n^2$ for $n \le 10^{8,000,000}$		
For	$\lvert\tan(n)\rvert/n^2$ is at most	
$1 \le n \le p_4$	$\lvert\tan(p_1)\rvert/(p_1)^2$	(≈ 1.56)
$p_4 \le n \le p_{118}$	$\lvert\tan(p_4)\rvert/(p_4)^2$	(≈ 1.87)
$p_{118} \le n \le p_{136}$	$\lvert\tan(p_{118})\rvert/(p_{118})^2$	$(\approx 4.14x10^{-59})$
$p_{136} \le n \le p_{315}$	$\lvert\tan(p_{136})\rvert/(p_{136})^2$	$(\approx 5.33x10^{-68})$
$p_{315} \le n \le p_{3727}$	$\lvert\tan(p_{315})\rvert/(p_{315})^2$	$(\approx 2.34x10^{-152})$
$p_{3727} \le n \le p_{3763}$	$\lvert\tan(p_{3727})\rvert/(p_{3727})^2$	$(\approx 1.48x10^{-1937})$
$p_{3763} \le n \le p_{15503}$	$\lvert\tan(p_{3763})\rvert/(p_{3763})^2$	$(\approx 1.90x10^{-1958})$
$p_{15503} \le n \le p_{153396}$	$\lvert\tan(p_{15503})\rvert/(p_{15503})^2$	$(\approx 3.50x10^{-7989})$
$p_{153396} \le n \le p_{156559}$	$\lvert\tan(p_{153396})\rvert/(p_{153396})^2$	$(\approx 2.51x10^{-78918})$
$p_{156559} \le n \le p_{984404}$	$\lvert\tan(p_{156559})\rvert/(p_{156559})^2$	$(\approx 1.27x10^{-80540})$
$p_{984404} \le n \le p_{1119377}$	$\lvert\tan(p_{984404})\rvert/(p_{984404})^2$	$(\approx 1.72x10^{-506884})$
$p_{1119377} \le n \le 10^{8,000,000}$	$\lvert\tan(p_{1119377})\rvert/(p_{1119377})^2$	$(\approx 1.92x10^{-576444})$

Although (4) has not been proven to converge, Rosenholtz has proven that (4) is very small for all $n \le 10^{8\,million}$. Rosenholtz further concludes if the limit of (4) does not equal 0, then it could be an addition to Guy's list of eventually failing patterns [**4**, p. 375].

Chapter 4: Laplace Transforms

One of the known methods for finding the exact values of convergent series involves Fourier analysis. Another method to find exact values is using Laplace transforms. With the help of Lesko and Smith, Efthimiou demonstrates how to find

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{x}{2}\right), \qquad 0 < x < 2\pi \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}, \qquad 0 < x < 2\pi \tag{6}$$

using Laplace transforms rather than Fourier analysis [**1**, p. 376]. Efthimiou feels the Fourier analysis approach requires more guesswork, while the Laplace transforms approach is more straightforward.

Efthimiou starts by explaining the method of finding the exact values of (5) and (6) and demonstrates the method employing an initial series as is detailed below. Efthimiou states that if one can find an explicit function

$$h(t) = \sum\nolimits_{n \in I} u_n e^{-nt} \,,$$

then this leads to a simple integral representation of the initial series:

$$\sum_{n \in I} u_n v_n = \int_0^{+\infty} h(t)\, f(t)\, dt \tag{7}$$

[**1**, p. 377].

Once the integration is performed, the expressions for sums of initial series are found.

To apply the method to trigonometric series, Efthimiou suggests using a specific form of the series where

$$S = \sum_{n \in I} \sin(nx) e^{-nt} \quad \text{and}$$

$$C = \sum_{n \in I} \cos(nx) e^{-nt}$$

[**1**, p. 377].

Efthimiou states that the summations are performed easily using complex notation:

$$C + iS = \sum_{n \in I} e^{inx} e^{-nt} .$$

In particular, when $I = N^* = \{1, 2, \ldots\}$, assuming that x is a real number and $t > 0$,

$$\sum_{n=1}^{\infty} \sin(nx) e^{-nt} = \frac{e^{-t} \sin x}{1 - 2\cos x e^{-t} + e^{-2t}} \tag{8}$$

$$\sum_{n=1}^{\infty} \cos(nx) e^{-nt} = \frac{e^{-t} (\cos x - e^{-t})}{1 - 2\cos x e^{-t} + e^{-2t}} \tag{9}$$

[**1**, p. 378].

Efthimiou uses a combination of (7), the Laplace transform approach, along with (8) and (9), and the complex notation of the trigonometric series to find the exact values of (5) and (6). Efthimiou begins with the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^v}$$

where $v \in N^*$ and $0 < x < 2\pi$, if $v = 1$, or $0 \le x \le 2\pi$, if $v > 1$ [**1**, p. 378]. Using (7) and

$$\frac{1}{n^v} = \frac{1}{(v-1)!} \int_0^{\infty} e^{-nt} t^{v-1} dt$$

the following equation is obtained:

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^{v}}=\frac{1}{(v-1)!}\sum_{n=1}^{\infty}\cos(nx)\int_{0}^{\infty}e^{-nt}t^{v-1}dt$$

[**1**, p. 378].

Knowing that the order of operations of summation and integration can be exchanged, Efthimiou obtains the next step of:

$$\frac{1}{(v-1)!}\int_{0}^{\infty}\left(\sum_{n=1}^{\infty}\cos(nx)\;e^{-nt}\right)t^{v-1}dt \tag{10}$$

[**1**, p. 378].

Substituting (9) into (10), Efthimiou finds

$$\frac{1}{(v-1)!}\int_{0}^{\infty}\frac{e^{-t}(\cos x-e^{-t})}{1-2\cos x\,e^{-t}+e^{-2t}}\;t^{v-1}dt$$

[**1**, p. 378].

Efthimiou then supposes $u=e^{-t}$, so the integral can be in the more compact form of

$$\frac{(-1)^{v-1}}{(v-1)!}\int_{0}^{1}\frac{\cos x-u}{1-2\cos xu+u^{2}}\;(\ln u)^{v-1}du$$

[**1**, p. 378].

For the special case of $v=1$,

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}=\frac{1}{2}\int_{0}^{1}\frac{d(1-2\cos xu+u^{2})}{1-2\cos xu+u^{2}}$$

[**1**, p. 379].

Using integral tables and calculations, finding the exact value of (5) does not require any guesswork. To find (6), Efthimiou follows similar steps. The integral representation for (6) is

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^{v}}=\frac{(-1)^{v-1}}{(v-1)!}\sin x\int_0^1\frac{(\ln u)^{v-1}}{1-2\cos xu+u^2}\,dt$$

[**1**, p. 379].

For the special case of $v=1$,

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\sin x\int_0^1\frac{1}{(u-\cos x)^2+\sin^2 x}\,du$$

[**1**, p. 379].

Using integral tables and calculations, (6) is found without any guessing as well.

Efthimiou suggests using Laplace transforms to find the exact values of (5) and (6) does not require any guessing and seems relatively easier than Fourier analysis. Efthimiou hopes “that readers will appreciate the ease and transparency of the method” [**1**, p. 379].

Chapter 5: Approximation of π

Another application of using trigonometric sequences and series can be used to find an approximation of π. Wallis' and Vieta's formulas can both help approximate π and are also surprisingly related. Wallis' formula is

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots\frac{2n\cdot 2n}{(2n-1)\cdot(2n+1)}\cdots,$$

and Vieta's formula is

$$\frac{\pi}{2}=\frac{2}{\sqrt{2}}\,\frac{2}{\sqrt{2+\sqrt{2}}}\,\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\,\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}+\sqrt{2}}}}\cdots$$

[**2**, p. 201].

Kreminski believes Vieta's formula is the better of the two because not only is Wallis' formula slow, Kreminski also found a way to accerelate Vieta's formula. More specifically, the accelerated version of Vieta's formula can accurately approximate π to over 300,000 digits [**2**, p. 201].

Kreminski proposes that Vieta's formula is easiest to derive. Vieta's formula uses trigonometric identities and the knowledge of $\lim\limits_{u\to 0}\frac{\sin u}{u}=1$ to derive

$$\begin{aligned}\sin(x) &= 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\\ &=2^2\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right)\\ &=2^3\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\sin\left(\frac{x}{8}\right)\\ &\cdots\end{aligned}$$

which leads to the conclusion that

$$\sin(x) = 2^n \left(\prod_{k=1}^{k=n} \cos\left(\frac{x}{2^k}\right) \right) \sin\left(\frac{x}{2^n}\right) \tag{11}$$

[**2**, p. 202].

After dividing each side of (11) by *x*, taking the limit as $n \to \infty$, setting $x = \frac{\pi}{2}$, and repeatedly applying the half-angle formula $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$, the reciprocal yields Vieta's formula.

Kreminski states that unlike the convergence of Wallis' formula, doubling the partial product of the first fifty terms of Vieta's formula approximates π accurate to about 30 digits [**2**, p. 202]. There is a slight error of $1.1 \cdot 10^{-30}$, so Kreminski bounds the error and accelerates the convergence of these partial products.

Kreminski then discusses the convergence and accelerated convergence of Vieta's formula. Kreminski defines p_n as the product of the first *n* terms. Kreminski observes when (11) is applied to $x = \frac{\pi}{2}$, $p_n = 2^n \sin\left(\frac{\pi}{2^{n+1}}\right)$. Therefore, using the Maclaurin series for sine,

$$\frac{\pi}{2} = p_n + \frac{k_1}{4^n} + \frac{k_2}{16^n} + \frac{k_3}{64^n} + \frac{k_4}{256^n} \ldots \quad \text{where } k_m = \frac{(-1)^{m-1}\pi^{2m+1}}{2^{2m+1}(2m+1)!} \tag{12}$$

[**2**, p. 203].

Kreminski uses (12) to develop the accelerated Vieta algorithm. Using a similar version of (12) and multiplying the version by 4 yields

$$4\frac{\pi}{2} = 4p_{n+1} + 4\frac{k_1}{4^{n+1}} + 4\frac{k_2}{16^{n+1}} + 4\frac{k_3}{64^{n+1}} + 4\frac{k_4}{236^{n+1}} \ldots \tag{13}$$

[**2**, p. 203].

After subtracting (12) from (13) and dividing the result by three,

$$\frac{\pi}{2} = \frac{4p_{n+1} - p_n}{3} + \frac{l_2}{16^n} + \frac{l_3}{64^n} + \frac{l_4}{256^n} + \ldots \tag{14}$$

where $l_j = ((4 - 4^j)k_j / (3 \cdot 4^j)$ for $j \geq 1$ [**2**, p. 203]. Then,

$$\frac{\pi}{2} = \frac{4p_{n+2} - p_{n+1}}{3} + \frac{l_2}{16^{n+1}} + \frac{l_3}{64^{n+1}} + \frac{l_4}{256^{n+1}} + \ldots . \tag{15}$$

By multiplying (15) by sixteen, subtracting from (14), and dividing the result by fifteen,

$$\frac{\pi}{2} = \frac{16r_{n-1} - r_n}{15} + \frac{m_3}{64^n} + \frac{m_4}{256^n} + \ldots \tag{16}$$

where $r_n = \frac{4p_{n+1} - p_n}{3}$ and $m_j = ((16 - 4^j)l_j / (15 \cdot 4^j)$ for $j \geq 2$ [**2**, p. 204].

By repeating the process over and over again, Kreminski finds that it produces a recursive sequence of approximations to $\frac{\pi}{2}$ that are increasingly accurate. Through numerical methods and a little experimentation as well as observation, Kreminski found an accelerated version of Vieta that can help approximate π accurately to over 300,000 digits.

Chapter 6: Conclusion

In conclusion, Sholander, Rosenholtz, Efthimiou, and Kreminski shed new light on the uses of trigonometric sequences and series. Sholander uses Fibonacci trigonometric double sequences to define the trigonometric functions sine and cosine. Rosenholtz's article takes tangent series and sequences to a new level by exploring theorems that are not in most Calculus books. With the help of Lesko and Smith, Efthimiou shows Laplace transforms can also help find exact values of trigonometric series. Kreminski uses trigonometric sequences and series to accurately approximate π to thousands of digits.

The National Council of Teachers in Mathematics suggests 9^{th}-12^{th} grade students should "understand how mathematical ideas interconnect and build on one another to produce a coherent whole" [**3**]. Students can build on Sholander's idea and use different mathematical concepts to define other functions. Sholander shows students how elementary mathematical concepts help build a foundation for upper level mathematical ideas such as trigonometric functions.

The National Council of Teachers in Mathematics also suggests 9^{th}-12^{th} grade students should "build new mathematical knowledge through problem solving" [**3**]. Despite the fact that most Calculus textbooks do not contain Rosenholtz's theorems, students can still explore the new ideas introduced in the article by drawing from previously covered material. Furthermore, Kreminski showed using numerical methods how to approximate π to thousands of digits. Although approximating π to a few digits may look difficult, Kreminski encourages students to problem solve and discover new things.

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Vita

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